Stochastic Programming, Cooperation, and Risk Exchange

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Abstract. Stochastic programming offers handy instruments to analyze exchange of goods and risks. Absent efficient markets for some of those items, such programming may imitate or synthesize market-like transfers among concerned parties. Specifically, using shadow prices (Lagrange multipliers) on aggregate endowments, one may identify side-payments that yield core solutions to cooperative production games.

1. Introduction

Economics depends on - and amply demonstrates - theoretical and practical advantages stemming from exchange of private goods. Similarly, insurance thrives and builds on mutual benefits derived from pooling and sharing private risks. Sometimes however, appropriate markets or institutions are not there to mediate desirable transactions. Then, as is well known, optimization methods, and notably duality theory, may help in assessing the value of potential exchanges. Less known is that optimization can also single out price-based transfers (or side-payments), these serving as surrogates for reasonable market-like deals.

The following example, first studied by Owen [14], illustrates such issues well: Suppose each agent i, belonging to a finite society I, faces a linear program

$$v^i := \max\left\{c \cdot x \;\middle|\; x \geq 0, b^i - Ax \geq 0\right\}, \tag{1}$$

assumed feasible, with finite value v^i. Here A is construed as a $m \times n$ activity matrix; the vector $b^i \in \mathbb{R}^m$ denotes i's bundle of m different resources; and finally, $c \in \mathbb{R}^n$ accounts for unit contributions created by activity plans $x \in \mathbb{R}^n_+$. Most often the said resources come in non-desirable proportions, causing shortages, excesses, or bottlenecks. Flexibility and gains can then be had by pooling the privately held endowments. Specifically, a coalition $S \subseteq I$, whose members altogether control the resource bundle $b^S := \sum_{i \in S} b^i$, could achieve an optimal value

$$v^S := \max\left\{c \cdot x \;\middle|\; x \geq 0, b^S - Ax \geq 0\right\} \tag{2}$$

which exceeds the individually assembled revenue: $v^S \geq \sum_{i \in S} v^i$. So, given advantages in aggregation, it is fitting to ask: *How can potential gains of cooperation be secured and split?* For a quick and motivating answer, suppose there is an optimal

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dual solution (a so-called Lagrange multiplier) $\bar{y}$ to problem (2) when $S = I$. That price vector $\bar{y}$ evaluates (marginal) resources for the grand coalition $S = I$. Therefore, quite naturally, let i be offered payment $u^i := \bar{y} \cdot b^i$ for handing his holding over to the cooperative enterprise. Will he accept that offer? Yes, most likely! In fact, as it turns out, since $\sum_{i \in I} u^i = v^I$, this payment scheme covers the bill. Moreover, since $\sum_{i \in S} u^i \geq v^S$ for all $S \subset I$, nobody has economic incentives to object.

While following Evstigneev, Flåm [4], and Sandsmark [17], Owen's model will be extended here to accommodate uncertainty, nonlinearity, and several stages.[1] That much generality notwithstanding, it happens, under convexity assumptions, that rather similar results remain valid. Intuitively, and as already noticed, resource pooling facilitates substitutions and mitigates bottlenecks. Present uncertainty, that operation becomes even more important: It reduces environmental vagaries; it smoothens the effects of individual ups and downs. Put differently: Pooling allows not only reallocation over activities (or production lines) but also across events (or states of the world). Granted convexity and strict feasibility of the aggregate program, I shall show that total payoff may efficiently be split to the rational protest of nobody. Indeed, a so-called *core* cooperative solution is then easily synthesized by means of a Lagrange multiplier.

Building blocks and arguments for that result are organized below as follows. Section 2 begins by reviewing that part of stochastic programming which comes into play. Section 3 goes on to define an underlying, transferable-utility, cooperative game in its reduced, characteristic form. Section 4 exhibits core solutions, and section 5 concludes with a brief look at cooperation over time.

Some motivation for this note stems from the fact that economists and programmers often seem opposed (or rather ambivalent) about what sorts of decision-making constitute proper domains to explore. A narrow view holds that economics mainly reduces to the study of markets, whereas optimization primarily concerns single-agent planning. A broader view, exemplified below, encompasses collective action, self-interested agents, optimization, and market-inspired contracts.

2. STOCHASTIC PROGRAMS

Planning under uncertainty is always construed here as an optimization problem of the following generic form [5], [11], [15], [16]:

Maximize a real-valued, transferable (monetary) payoff $f_0(x) = f_0(x(\cdot))$ over suitable, finite trajectories $x = x(\cdot) = (x_1(\cdot), \ldots, x_T(\cdot))$ of random vectors $x_t(\omega) \in \mathbb{R}^{n_t}$. These vectors represent constrained choices - made sequentially, one at each stage or time $t = 1, 2, \ldots, T$ $(< \infty)$ - under imperfect knowledge about the scenario or state $\omega \in \Omega$ of the world. Although ω cannot be fully identified a priori, its probability distribution P is here presumed known; it is given exogenously and defined on some *sigma-field* $\mathcal{F}_{T+1}$ in Ω. Identification of ω improves over time. Specifically, there is an expanding family $\mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_T \subseteq \mathcal{F}_{T+1}$ of sigma-fields - or an unfolding scenario

[1] Related studies include [7], [8], [9], [10], [18].

tree - which describes the information flow. At time t one may ascertain for any event in $\mathcal{F}_t$ - and such events only - whether it has happened or not. In particular, a finite $\mathcal{F}_t$ would partition Ω into minimal events (atoms, information sets, decision nodes) pertaining to time t. The inclusion $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$, $t \le T$, which reflects progressive acquisition of knowledge, says that the said partition becomes finer as time evolves.

At time t the decision-maker implements the part x_t of his overall plan $x = (x_1, \ldots, x_T)$. That part is supposed to be a $\mathcal{F}_t$-measurable strategy (policy, behavioral rule) $x_t : \Omega \to \mathbb{R}^{n_t}$. This means that only available information is used at any stage; decisions must be based on observable, realized events. If so, the process $x = (x_1, \ldots, x_T)$ is declared *adapted* to the filtration $(\mathcal{F}_t)_{t=1}^T$. For example, let $\theta_1, \ldots, \theta_T$ be a stochastic process, defined on Ω, and let $\mathcal{F}_t$ be generated by $\theta_1, \ldots, \theta_t$. Then $\mathcal{F}_t$-measurability of x_t means that it depends on no more than $\theta_1, \ldots, \theta_t$. *Henceforth only adapted processes are considered.*

A mathematical issue crops up here, namely: Where does item x_t reside? Candidate addresses would be in spaces $L^{p_t}(\mathcal{F}_t, P; \mathbb{R}^{n_t})$ for suitable $p_t \in [1, +\infty]$. Being concerned below with modelling, and motivated by computation, I shall shy away from these technicalities and simply *assume that Ω be finite.* Then all spaces $L^{p_t}(\mathcal{F}_t, P; \mathbb{R}^{n_t})$, $p_t \in [1, +\infty]$, are topologically equal, and each is regarded as finite-dimensional Euclidean. (Nonetheless, the presentation below opens for extensions to infinite Ω.)

Besides informational limitations, and the insistence on adaptive processes, there are other restrictions, one being that

$$x_t(\omega) \in X_t(\omega) \text{ almost surely for each } t. \tag{3}$$

Here $\omega \rightsquigarrow X_t(\omega) \subseteq \mathbb{R}^{n_t}$ is a nonempty closed $\mathcal{F}_t$-measurable random set; see e.g. [1]. (For notational simplicity all inclusions that involve random objects are henceforth tacitly understood to hold almost surely. Similarly, all equalities or inequalities between random vectors, mentioned below, hold with probability one and componentwise.) Added to set-constraint (3) comes a family of explicit, functional constraints:

$$f_{1t}(\omega, x) := f_{1t}(\omega, x_1(\omega), \ldots, x_t(\omega)) \in \mathbb{R}_+^{m_t} \text{ for all } t, \tag{4}$$

this inclusion featuring a vector-valued function f_{1t} which is $\mathcal{F}_t \times$Borel-measurable. For short I write $x \in X$ and $\mathbf{f}_1(x) \ge 0$ to indicate satisfaction of (3) and (4), respectively. Planning under uncertainty is now formalized succinctly as problem

$$(P): \quad \sup \{ f_0(x) \mid x \in X \text{ and } \mathbf{f}_1(x) \ge 0 \},$$

assumed feasible with finite optimal value, denoted $\sup(P)$. To open up for duality let

$$\mathbf{R}_1 := \{ \mathbf{r}_1 = (r_{1t}) \mid r_{1t} : \Omega \to \mathbb{R}^{m_t} \text{ is } \mathcal{F}_t\text{-measurable}, \forall t \}$$

denote the (canonical) space of *additive right-hand-side perturbations* in $\mathbf{f}_1(x) \ge 0$. Any member $\mathbf{r}_1$ of $\mathbf{R}_1$ transforms the latter inequality into $\mathbf{f}_1(x) \ge \mathbf{r}_1$. Correspondingly, define

$$Y := \left\{ y = (y_t) \;\middle|\; y_t : \Omega \to \mathbb{R}_+^{m_t} \text{ is } \mathcal{F}_t\text{-measurable}, \forall t \right\}$$

as the non-negative *cone of adapted multipliers.* Since a special sort, called *Lagrange multipliers,* will be crucial in the sequel, an existence result (Theorem 1) is provided shortly. For the statement let $\mathbb{E}$ denote the expectation operator with respect to probability measure P.

Proposition 1 (A Fritz John multiplier rule). *Suppose problem* (P) *has finite optimal value* $\sup(P)$. *Suppose also that the convex hull of the set*

$$C := \{(r_0, \mathbf{r}_1) \in \mathbb{R}\times\mathbf{R}_1 : f_0(x) - \sup(P) \geq r_0,\ \mathbf{f}_1(x) \geq \mathbf{r}_1 \text{ for some } x \in X\}$$

has $(0, \mathbf{0})$ *at its boundary. Then there exists a nonzero* $(r_0^*, \mathbf{r}_1^*) \in \mathbb{R}_+ \times Y$ *such that*

$$\sup\{r_0^* f_0(x) + \mathbb{E}[\mathbf{r}_1^* \cdot \mathbf{f}_1(x)] : x \in X\} = r_0^* \sup(P). \tag{5}$$

Proof. Through the boundary point $(0, \mathbf{0})$ of C there passes, by assumption, a closed supporting hyperplane. Hence some nonzero $(r_0^*, \mathbf{r}_1^*) \in \mathbb{R}\times Y$ defines a continuous linear functional $(r_0, \mathbf{r}_1) \mapsto r_0^* r_0 + \mathbb{E}[\mathbf{r}_1^* \cdot \mathbf{r}_1]$ which is ≤ 0 on C. (Were not Ω finite, then additional assumptions might be needed to justify these claims.) Clearly,

$$(r_0, \mathbf{r}_1) \in C \ \&\ (r_0, \mathbf{r}_1) \geq (r_0', \mathbf{r}_1') \in \mathbb{R}\times\mathbf{R}_1 \Rightarrow (r_0', \mathbf{r}_1') \in C.$$

Therefore, $(r_0^*, \mathbf{r}_1^*) \gneqq 0$. Since $[f_0(x) - \sup(P), \mathbf{f}_1(x)] \in C$ whenever $x \in X$, it follows that $r_0^*[f_0(x) - \sup(P)] + \mathbb{E}[\mathbf{r}_1^* \cdot \mathbf{f}_1(x)] \leq 0$ for all $x \in X$, this inequality implying

$$\sup\{r_0^* f_0(x) + \mathbb{E}[\mathbf{r}_1^* \cdot \mathbf{f}_1(x)] : x \in X\} \leq r_0^* \sup(P).$$

The reverse inequality follows from $\mathbf{r}_1^* \cdot \mathbf{f}_1(x) \geq 0$ whenever x is feasible. $\square$

Convexity played a key role in Proposition 1. So, some arguments in support of that property are called for. To that end, suppose $\mathbf{f} := (f_0, \mathbf{f}_1)$ is *concave-like* on X. This means that for any two points $x', x'' \in X$ and any number $\rho \in (0, 1)$, there should exist a third point $x \in X$ such that $\mathbf{f}(x) \geq \rho\mathbf{f}(x') + (1 - \rho)\mathbf{f}(x'')$. Under that assumption C becomes convex, and then $(0, \mathbf{0})$ belongs to its boundary. This observation shows that (5) is rather well motivated - and especially so for convex instances. Associate now to problem (P) its "ordinary" *Lagrangian*

$$(x, y) \in X \times Y \mapsto L(x, y) := f_0(x) + \mathbb{E}[y \cdot \mathbf{f}_1(x)].$$

Theorem 1 (Normal Lagrange multipliers). *Assume* (5) *and the following strict feasibility condition: for any right-hand-side perturbation* $\mathbf{r}_1 \in \mathbf{R}_1$ *that belongs to some open set containing* $\mathbf{0}$, *one can find* $x \in X$ *satisfying* $\mathbf{f}_1(x) \geq \mathbf{r}_1$. *Then then there exists a multiplier vector* $y \in Y$ *such that* $\sup(P) = \sup_{x\in X} L(x, y)$.

Proof. Let V denote a vicinity of $\mathbf{0}$ such that for any $\mathbf{r}_1 \in V$ some $x \in X$ satisfies $\mathbf{f}_1(x) \geq \mathbf{r}_1$. Suppose $r_0^* = 0$. Then $\mathbf{r}_1^* \gneqq 0$, and we may choose a positive real number r_1 so large that $\mathbf{r}_1 := \mathbf{r}_1^*/r_1 \in V$. By assumption there exists $x \in X$ such that $\mathbf{f}_1(x) \geq \mathbf{r}_1 \gneqq 0$, whence

$$0 < \sup\left\{\mathbb{E}\left[\mathbf{r}_1^* \cdot \mathbf{f}_1(x)\right] : x \in X\right\} = r_0^* \sup(P) = 0.$$

This contradiction proves r_0^* strictly positive. So, in (5) divide through by r_0^* and let $y := \mathbf{r}_1^*/r_0^*$ to have the desired conclusion. □

The following case is commonly known: If X is convex, and $\mathbf{f}_1$ is concave, then the *Slater condition* (that some adapted $x \in X$ yields strict feasibility: $\mathbf{f}_1(x) > 0$) implies $r_0 > 0$.

3. Cooperation and Risk Exchange

There is a circumscribed, fixed, finite set I of individual agents, each more or less plagued by resource scarcity, risk, and technological hurdles. Specifically, individual $i \in I$ faces a stochastic program

$$v^i := \sup\left\{f_0^i(x^i) : x^i \in X^i \text{ and } \mathbf{f}_1^i(x^i) \geq 0\right\}$$

of the sort (P^i) described in Section 2. As there, the objective $f_0^i(x^i)$ denotes a monetary amount, perfectly divisible and transferable. It must be stressed that many items mentioned in Section 2 are common and publicly known, namely: the time horizon T, the information flow $(\mathcal{F}_t)$, and the probability space $(\Omega, \mathcal{F}_{T+1}, P)$.

Quite as above individual i is constrained in two important ways: At each time $t = 1, \ldots, T$ his decisions must satisfy

$$x_t^i(\omega) \in X_t^i(\omega) \subseteq \mathbb{R}^{n_t^i} \quad \text{and} \quad f_{1t}^i(\omega, x_1^i(\omega), \ldots, x_t^i(\omega)) \in \mathbb{R}_+^{m_t}.$$

These constraints involve sets $\omega \rightsquigarrow X_t^i(\omega)$ and functions $(\omega, x^i) \mapsto f_{1t}^i(\omega, x_1^i(\omega), \ldots, x_t^i(\omega))$ that are $\mathcal{F}_t$-measurable. Note that the basic decision spaces $\mathbb{R}^{n_t^i}$ can vary across agents (and time), but most important: all functions $f_{1t}^i, i \in I$, have the same image space $\mathbb{R}^{m_t}$. In this setting a coalition $S \subseteq I$ could achieve stand-alone value

$$v^S := \sup\left\{\sum_{i\in S} f_0^i(x^i) : x^i \in X^i, \forall i \in S, \text{ and } \sum_{i\in S}\mathbf{f}_1^i(x^i) \geq 0\right\}.$$

Whether that optimal value is computed or not, I tacitly assume, somewhat heroically, that no agent i misrepresents privately held data to own advantage. (Market games with *differential information* have been considered in [2], [3], [19].)

For cooperation to comprise everybody it should leave no individual - and no coalition - worse off than alone. In other words: the concerting of actions, and the joining of forces, requires satisfaction of numerous participation constraints. To that end money transfers (compensations or "bribes") may certainly help. These should reasonably produce a *payoff allocation* $u = (u_i) \in \mathbb{R}^I$ that entails

$$\begin{array}{lrcl} \textit{Pareto efficiency:} & \sum_{i\in I} u^i & = & v^I, \\ \textit{and stability:} & \sum_{i\in S} u^i & \geq & v^S \quad \textit{for all coalitions } S \subset I. \end{array} \tag{6}$$

Stability means here that no singleton or set $S \subset I$ of several players could improve their outcome by splitting away from the society.[2] Note that mere stability is easy to achieve: Simply let the numbers u^i be so large that $\sum_{i\in S} u^i \geq v^S, \forall S \subseteq I$. Therefore, the essential difficulty resides in the efficiency requirement. I ask: *Can* (6) *be solved? If so, how?* These questions fit the frames of a (payoff-sharing) cooperative game with *player set* I, *characteristic function* $I \supseteq S \mapsto v^S \in \mathbb{R} \cup \{-\infty\}$, and *side payments.* Any solution $u = (u^i) \in \mathbb{R}^I$ to (6) is said to be an element in the *core.*

4. CORE SOLUTIONS

Write $x = (x^i) = (x^i_t)$ and consider the Lagrangian

$$L^S(x,y) := \begin{cases} \sum_{i\in S}\left\{f_0^i(x^i) + \mathbb{E}\left[y\cdot \mathbf{f}_1^i(x^i)\right]\right\} & \text{if } x^i \in X^i \text{ for all } i \in S, \\ -\infty & \text{otherwise} \end{cases}$$

associated to coalition S. As customary, $v^S = \sup_x \inf_{y\in Y} L^S(x,y) \leq \sup_x L^S(x,y)$ for all $y \in Y$. Therefore I declare $\bar{y} \in Y$ *a Lagrange multiplier for the grand coalition* iff the reverse inequality hold when $S = I$, i.e. if $\sup_x L^I(x,\bar{y}) \leq v^I$. Existence of such a multiplier is guaranteed under conditions stated in Theorem 1. The next result shows that any Lagrange multiplier may incite cooperation:

Theorem 2 (Lagrange multipliers yield core solutions). *For any Lagrange multiplier $\bar{y}$ of the grand coalition the payoff allocation*

$$u^i := \sup\left\{f_0^i(x^i) + \mathbb{E}\left[\bar{y}\cdot \mathbf{f}_1^i(x^i)\right] : x^i \in X^i\right\},\ i \in I,$$

belongs to the core.

Proof. Stability obtains because any coalition $S \subseteq I$ receives

$$\sum_{i\in S} u^i = \sup_x L^S(x,\bar{y}) \geq \inf_{y\in Y}\sup_x L^S(x,y) \geq \sup_x \inf_{y\in Y} L^S(x,y) = v^S, \tag{7}$$

the very last inequality often being referred to as *weak duality.* The hypothesis concerning $\bar{y}$ ensures *strong duality.* Indeed, that hypothesis yields $\sum_{i\in I} u^i = \sup_x L^I(x,\bar{y}) \leq v^I$. The upshot is that Pareto efficiency also prevails. □

A slightly different approach helps to understand and supplement Theorem 2. In view of (1) and (4) let the function

$$f^i_{1t}(\omega, x^i_1, \ldots, x^i_t) = b^i_t(\omega) - A^i_t(\omega, x^i_1, \ldots, x^i_t) \in \mathbb{R}^{m_t}_+,$$

explicitly incorporate random resources $b^i_t(\omega) \in \mathbb{R}^{m_t}$ as well as an operator $A_i : \Omega \times \mathbb{R}^{n^i_1+\cdots+n^i_t} \to \mathbb{R}^{m_t}$. Define then

$$\pi^i(b^i) := \sup\left\{f_0^i(x^i) \;\middle|\; x^i_t \in X^i_t \text{ and } b^i_t - A^i_t(x^i_1, \ldots, x^i_t) \geq 0 \text{ for all } t\right\},$$

[2] Coalitions are here *orthogonal* in the sense that members of S can jointly obtain v^S regardless of what the outsiders $i \in I \setminus S$ undertake.

and consider the program $\sup\left\{\sum_{i\in I}\pi^i(z^i) \mid \sum_{i\in I} z^i = \sum_{i\in I} b^i\right\}$. Let $\bar{y}$ be a Lagrange multiplier of the latter. Then the allocation

$$u^i := \mathbb{E}\left[\bar{y}\cdot b^i\right] + \sup\left\{\pi^i(z^i) - \mathbb{E}\left[\bar{y}\cdot z^i\right] \mid z^i \text{ adapted}\right\} \tag{8}$$

belongs to the core. Evidently, formula (8) generalizes Owen's result, introduced in Section 1. More precisely: if the resource bundles b^i mentioned there are random, then such endowment commands core payment $u^i = \mathbb{E}\left[\bar{y}\cdot b^i\right]$ to its owner. The much studied instance of two-stage linear programs is particularly informative. That instance posits sure resource availability $b_1^i \in \mathbb{R}^{m_1}$ right now and random supply $\omega \mapsto b_2^i(\omega) \in \mathbb{R}^{m_2}$ next period. Consequently, it furnishes cooperative payoff

$$u^i = \bar{y}_1 \cdot b_1^i + \mathbb{E}\left[\bar{y}_2 \cdot b_2^i\right] \tag{9}$$

to agent i. Formula (9) brings out several noteworthy features: First, if second-stage endowments are correlated across agents, then most likely $\mathbb{E}\left[\bar{y}_2 \cdot b_2^i\right] \neq \mathbb{E}\bar{y}_2 \cdot \mathbb{E}b_2^i$. Thus, as in finance - and notably within the capital asset pricing model - covariance between a "paper" or "security" b^i and the aggregate (entire market portfolio) b^I becomes decisive for its pricing [11]. Intuitively, the expected second-period payment $\mathbb{E}\left[\bar{y}_2 \cdot b_2^i\right]$ to agent i depends then on two things: partly, on his *average contribution* $\mathbb{E}b_2^i$, and partly, on how his *realized contribution* $b_2^i(\omega)$ co-varies with the total second-stage endowment $b_2^I(\omega)$. To see this fact most simply, suppose a single resource comes into play at the second stage. Then $\mathbb{E}\left[\bar{y}_2 \cdot b_2^i\right] = \mathbb{E}\bar{y}_2 \cdot \mathbb{E}b_2^i + cov(\bar{y}_2, b_2^i)$. Thus an agent who brings much of a resource when it is scarce, will insure his co-players and thereby receive handsome compensation.

Another speaking property of (9) is the step-wise, separable nature of payment. The next section concludes with a more general view at this property.

5. COOPERATION OVER TIME

It is fitting to elaborate on the dynamics of cooperation. For that purpose write $x_{[1,t]}^i := (x_1^i, \ldots, x_t^i)$ and assume here time-separable, expected payoff in the form $f_0^i(x^i) := \mathbb{E}\sum_{t=1}^T f_{0t}^i(\omega, x_t^i(\omega))$. Correspondingly, coalition $S \subseteq I$ would now deal with the Lagrangian

$$L^S(x,y) := \sum_{i\in S}\sum_{t=1}^{T}\mathbb{E}\left[f_{0t}^i(\omega, x_t^i(\omega)) + y_t(\omega)\cdot f_{1t}^i(\omega, x_{[1,t]}^i(\omega))\right].$$

Theorem 3 (Multistage core elements). *Suppose $\bar{y}$ is a Lagrange multiplier for the grand coalition. Then the payoff allocation*

$$u^i := \sup_{x^i\in X^i}\sum_{t=0}^{T}\mathbb{E}\left[f_{0t}^i(\omega, x_t^i(\omega)) + \bar{y}_t(\omega)\cdot f_{1t}^i(\omega, x_{[1,t]}^i(\omega))\right]$$

belongs to the core. Moreover, for any interim time $t < T$, featuring sunk but optimal decisions $x_{[1,t]} = (x^i_{[1,t]})_{i \in I}$, the remaining cooperative enterprise admits a conditional core allocation

$$u^i_t(x_{[1,t]}) := \sup_{x^i_\tau \in X^i_\tau, \tau > t} \sum_{\tau > t} \mathbb{E}\left[f^i_{0\tau}(\omega, x^i_\tau(\omega)) + \bar{y}_\tau(\omega) \cdot f^i_{1\tau}(\omega, x^i_{[1,\tau]}(\omega)) \,\middle|\, \mathcal{F}_t, x_{[1,t]} \right]. \quad \Box \tag{10}$$

This result points to the sequential consistency of allocation along the realized path. That is, the payment scheme u, when stated in terms of contingent transfers (10), will resist re-negotiation over time and events. Can contracts of that sort come into existence? Extensive fieldwork often see agents who voluntarily organize themselves to secure the benefits of trade and mutual risk protection [13]. We stress though that endowments were private here; the issue was not provision of public goods [12]. Potential application to trade in greenhouse gases is outlined in [6].

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